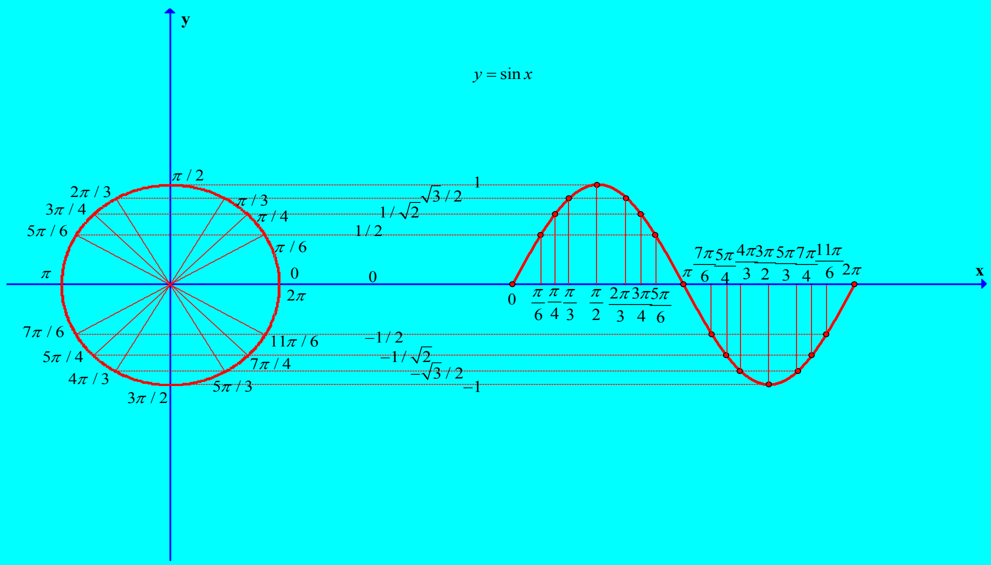
Concept based and Exam oriented

TUTORIAL LESSONS

Mathematical Methods
1 & 2



In This Book

Fully worked solutions to all questions

Written explanation of theory and concepts for each question

Over 1000 specifically designed study and exam style questions

Strong focus on algebra from basic to the highest complexity

Answers to most of your questions

Help for homework and assessment

Exam preparation through each lesson

Systematic revision while study new topics

B. Z.

Disclaimers

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CAPITAL TUITION

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About the book

Even though this book is fully compatible with the Victorian school Curriculum and contains all important topics and can also be very useful as supplementary study material in other states, the book is still different than any school text book as well as from most of the other mathematics books because all questions in this book are fully worked out and concisely explained in full details.

This book is designed to help you to find answers to, almost, all of your questions. Written by tutor with over 30 years tutoring experience, the book contains over 1000 fully worked questions with concise theoretical and conceptual explanation for each question. The intellectual base for this book is a systematic collection of questions asked by students over a long period of time. In order to provide students with all subject requirements the author created questions by adding other mathematical aspects such as: plenty algebra, application of mathematical theory and development of mathematical concepts and all this with strong tests and exam orientation. Finally, the wide range of different style of questions is helping students to learn mathematics with deeper understanding of theory and concepts that can be applied across the whole range of questions, rather than trying to apply some memorized patterns that usually can work on some certain questions. So this book will, definitely, help you:

- To find answers to most of your questions (even the most complex once)
- To do your homework and school tasks with easy and satisfactory way
- To be ready for your school assessments and final year exam
- To improve algebra skills
- To develop independent studying skills
- To get knowledge of the highest standard in the fastest time
- To understand and memorise even the most complex concepts in an easy way because most of the concepts are broken into simple steps and constantly revised throughout the book.
- To develop a high ability to apply theoretical knowledge and concepts in solving questions.
- To extend your educational ability and to increase motivation as you will discover that “knowledge is the power” and studying is not such hard work.

The book contains all important topics for Mathematical Methods 1&2, such as:

- Linear Equation and Graph
- Quadratic Equations and Functions
- Cubic Polynomial, Equations and Functions
- Functions, Relations and Transformation of Graphs
- Exponential and Logarithmic Equations and Functions
- Trigonometry
- Differentiation and Integration. Application

There are 40 tutorial lessons. Each lesson (session), contains in average about 25 fully worked out questions with concise explanation of theory and concept applied for each question. The firs (or first few) sessions of each topic were designed to present theory and basic concepts for that topic. The other sessions are practices sessions giving to students the best examples to get full understanding of the subject. Questions are not arranged according to their level of complexity, because all questions are fully worked out and concisely explained so that every student (regardless of current level of knowledge) can easily understand algebra, concept and theory applied on each questions. The other reason why questions are not graded is to motivate students to work across the entire book rather than to work selectively. Such arrangement and very detailed explanation of questions enable to students of all levels of knowledge to progress with this book to the highest possible standard.

Finally, this book is an attempt to save student's time and money seeing tutors around and to make available to everyone the best tutorial lessons that has been delivered to students over a few decades.

A short author’s message about how to study Mathematics

Choosing the right way of doing something is very solid base for success. As studying is a complex activity, so we have to be more conscious of choosing the right way of studying. This way is, generally, different from subject to subject and as Mathematics is one of very specific and quite complex subject, we have to take the right approach to that subject to ensure our success.

Definitely, we can’t study Mathematics by reading even not by memorising simple facts and formulas. Studying Mathematics requires permanent practice over long period. As we are progressing from grade to grade, we have to keep fresh our previous knowledge (theory, concepts, even facts and formulas) and to build up that knowledge from current program.

When we start studying a new chapter, of course, we have to “read” the book. During “reading” time we have to stop “reading” whenever is necessary to write down important words, facts and formulas or to draw a graph or some other appropriate drawing, even if that graph or drawing is already in the book. During that time we have to identify the meaning of new terminology, maybe some formulas and definitions, key facts, theoretical explanations, probably some new concepts and finally to identify previously learned knowledge we should apply in the current chapter. When you are happy with your understanding of what you “read” then you can move to the next phase of studying. That is practicing. But how someone can be sure that his/her understanding is good enough to move to practicing? Here I wouldn’t suggest any quiz or simple test questions trial. Instead, I would recommend a different technique. That is to visualise in front of you your friend, cousin or your parents and try to teach them of what you have been learning during “reading” time. If necessary, go back in book to refresh your memory or to check if you have missed something. The theory of each chapter of most of the books is usually written on just a few pages, so that “reading” process shouldn’t consume long time. One or two hours should qualify you to move to practicing phase.

Once when you start practicing, you can start from easier questions and go to the most complex once. For each question, before you start working, spend few minutes time to think about an appropriate concept and draw a simple strategy how to apply that concept. When you start work make sure that your work is inside your concept and strategy drawn at the beginning. If the concept, you have, doesn’t work for that question you should try to find out the clear reason why that concept is not good. That will help you to correct the concept and satisfactory to finish the question.

When due to move on another chapter, you should select some testing questions and to conduct short test of your understanding of current chapter. It is very useful, at the end of studying the chapter, once again, pretend to be a teacher and try to teach somebody. Compare your way of “teaching” somebody after “reading” phase and after practising phase of the end of studying the chapter and you will discover how big progress you made.

How to use this book

Thinking about method of how to study Mathematics, you already may have some ideas how to use this book. Even though you can use this book independently from any other books, still it would be better to study the chapter in your school and when you already have some knowledge of the chapter then to start using this book. Go over first session of the chapter to establish solid understanding of theory and important concepts. Even though you have all questions fully worked out, I am suggesting you to try to solve questions before checking working solution. After each question even if you solve the question correctly, go through worked solution with high attention as always there will be something there that you may learn. It is, also, recommended when going through worked solutions, to frequently reefer to theoretically session(s) to learn how the author is applying theory and concepts on practical questions.

Let start

Good luck

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TRIGONOMETRIC FUNCTIONS

Notes & Study Examples

Sketching graphs of trigonometric functions

The method is transformations of graphs. This is the same method we studied in semester one. So we have to memorize the basic graphs and their features and to understand standard formula for transformation. The standard formula is

$$y = \overset{\overset{1}{\uparrow}}{\pm}\underset{\underset{2}{\downarrow}}{a}\sin\left(\overset{\overset{3}{\uparrow}}{\pm}\underset{\underset{4}{\downarrow}}{n}\left(x \underset{\underset{5}{\downarrow}}{\pm h}\right)\right)\underset{\underset{6}{\downarrow}}{\pm k} \qquad \text{where}$$

1 and 3 $\longrightarrow$ are reflections in x axis.

Note: for $y=\sin x$ if $\left.\begin{cases} y=-\sin x \\ y=\sin(-x) \end{cases}\right\} \Rightarrow$ there is reflection

If $y=-\sin(-x) \Rightarrow$ no reflection

For $y=\cos x$ if $\left.\begin{cases} y=-\cos x \\ y=-\cos(-x) \end{cases}\right\} \Rightarrow$ there is reflection

If $y=\cos(-x) \Rightarrow$ no reflection

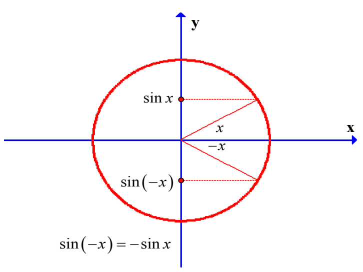
2 $\longrightarrow$ a is amplitude (dilation from x axis)

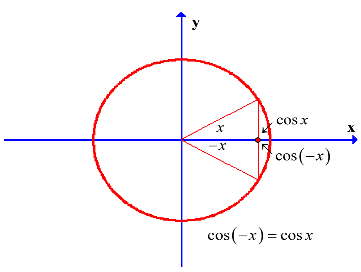
4 $\longrightarrow$ n is the number to calculate period (dilation from y). $T=\dfrac{2\pi}{n}$ for $y=\sin x$ and $y=\cos x$. But $T=\dfrac{\pi}{n}$ for $y=\tan x$

5 $\longrightarrow$ h is horizontal translation (if – to the right, if + to the left)

6 $\longrightarrow$ k is vertical translation (if – down, if + up)

The above standard formula is given for $y=\sin x$, but it can be used for $y=\cos x$ and $y=\tan x$. The only difference is in reflection for $y=\sin x$ and $y=\cos x$ (see reflection) and period for $y=\sin x$ and $y=\cos x$ is 2π while for $y=\tan x$ is π. The difference in reflections is because $\sin(-x)=-\sin x$ but $\cos(-x)=\cos x$. We can se this from unit circle.





Steps to sketch Trigonometric functions graphs

1) Check if your function is in its standard form. If not, change it

2) Look for vertical translation and draw horizontal dotted line representing vertical translation. Note. This line will be middle line of your graph.

3) Look for amplitude and from the middle line go up and down for number of units of amplitude. You may draw another two dotted lines and they will be the range of your graph.

4) Calculate period applying formula $T = \frac{2\pi}{n}$ and from origin mark the end point of period on x axis. Divide the length of period in four equal parts and calculate those points and mark them on x axis. Those four points correspond to $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π for the basic graph. Now bring your points (points you calculated) on your middle line.

5) Check for reflection. Note: Trigonometric functions have reflection only in the x axis. Now mark all critical points of your graph on the lines from steps 2 and 3 by following pattern of the corresponding basic graph. At the moment ignore horizontal translation if there is any. Alternatively you may sketch graph without horizontal translation by dotted line.

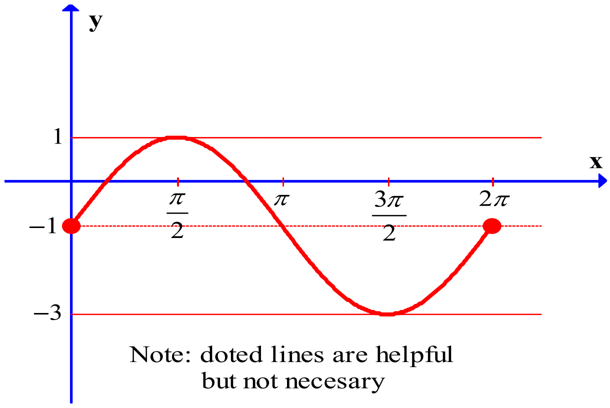
6) Check for horizontal translation and if there is any then move horizontally each of points obtained in step 5 in the direction and the length as indicated by horizontal translation. By connecting those points you will finish one full cycle of your graph. Now check for domain required for the graph and extend or cut your graph as it should be placed only in required domain. Do not forget to write coordinates of end points of your graph if those points are not the same as critical points. You don't need to mark x-int. if they are not required by the question.

Note: There is a small difference for sketching $y = \tan x$. Step 3 doesn't give us range and could be, even, ignored. Steps 4 and 6 are the most important as we have to determine equations of vertical asymptotes. You will need, for this, to take care for correct placement of one full cycle of basic graph inside of period that you calculated. Calculate period by formula $T = \frac{\pi}{n}$

Examples Sketch one full cycle of the following graphs

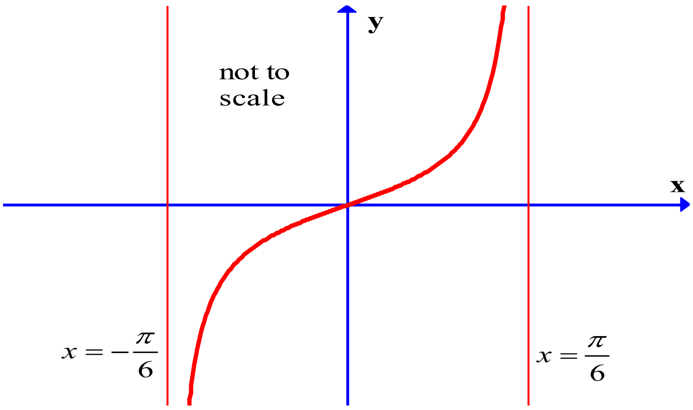
1. $y = 2\sin x - 1$

- Vertical translation is 1 unit down
- Amplitude is 2
- Period is 2π
- No reflection
- No horizontal translation



2. $y = \tan 3x$

- No vertical translation
- Amplitude (dilation) is 1
- Period is $\frac{\pi}{3}$; $\left(T = \frac{\pi}{3}\right)$
- No reflection
- No horizontal translation
- Asymptote $x = \pm\frac{\pi}{6}$



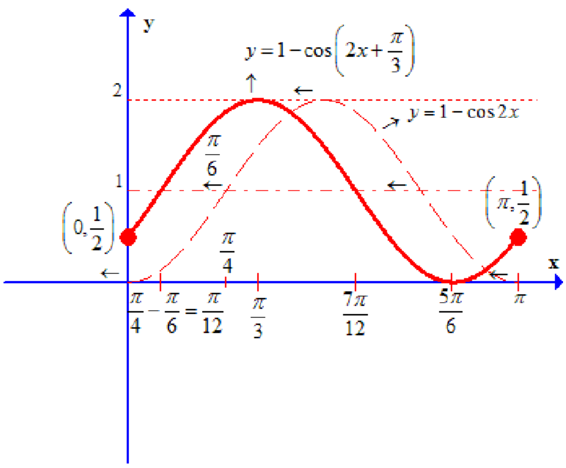
Note; We can draw rule (formula) for asymptote. That is $x = \pm\frac{T}{2}$, if there is no horizontal translation, where T is period. If we have horizontal translations then translate $x = \pm\frac{T}{2}$ in direction and magnitude of h. Take care for domain.

3. $y = 1 - \cos\left(2x + \frac{\pi}{3}\right)$

- Not in standard form Change it

$y = 1 - \cos\left[2\left(x + \frac{\pi}{6}\right)\right]$

- Vertical translation is 1 unit up
- Amplitude is 1
- Period is π ; $\left(T = \frac{2\pi}{2}\right)$
- Reflected, and Horizontal translation of $\frac{\pi}{6}$ units to the left

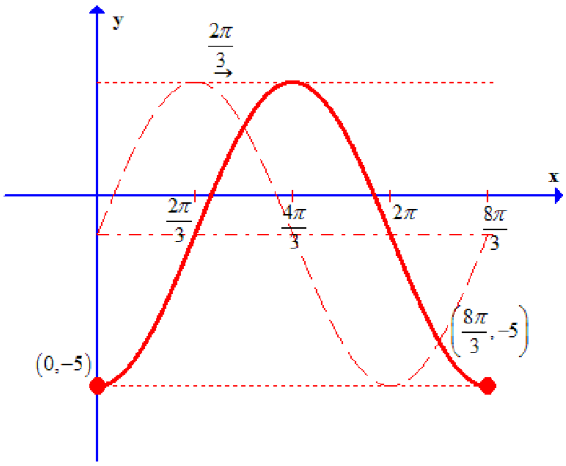


4. $y = 4\sin\left(\frac{3x}{4} - \frac{\pi}{2}\right) - 1$

- Not in standard form Change it

$y = 4\sin\left[\frac{3}{4}\left(x - \frac{2\pi}{3}\right)\right] - 1$

- Vertical translation is 1 unit down
- Amplitude is 4
- Period is $\frac{8\pi}{3}$; $\left(T = \frac{2\pi}{\frac{3}{4}} = \frac{8\pi}{3}\right)$
- No reflection, but Horizontal translation is $\frac{2\pi}{3}$ to the right

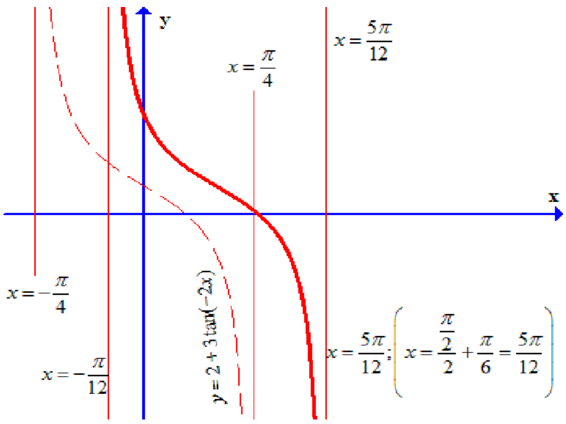


5. $y = 2 + 3\tan\left(\frac{\pi}{3} - 2x\right)$

- not in standard form. Change it

$y = 2 + 3\tan\left[-2\left(x - \frac{\pi}{6}\right)\right]$

- Vertical translation of 2 units up
- Amplitude (dilation) is 3
- Period is $\frac{\pi}{2}$. Reflected
- Horizontal translation is $\frac{\pi}{6}$ right.



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12.

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TRIGONOMETRIC EQUATIONS

Notes & Study Examples

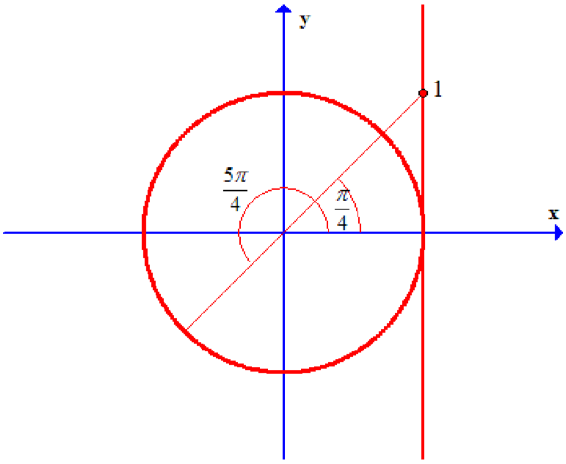
Examples

10) $\sin \boldsymbol{x} - \cos \boldsymbol{x} = 0 \qquad \boldsymbol{x} \in [0, 2\pi]$

$\sin \boldsymbol{x} = \cos \boldsymbol{x} \ / \div \cos \boldsymbol{x}$

$\frac{\sin \boldsymbol{x}}{\cos \boldsymbol{x}} = \frac{\cos \boldsymbol{x}}{\cos \boldsymbol{x}}$

$\tan \boldsymbol{x} = 1$



Again we have two different functions but now we can't apply null factor law. All function must be the same ($\sin \boldsymbol{x}$, $\cos \boldsymbol{x}$ or $\tan \boldsymbol{x}$). The best way to do that will be to rearrange and divide equation by $\cos \boldsymbol{x}$ because we can use formula $\tan x = \frac{\sin x}{\cos x}$

Now we have the simplest form and we will continue solving this equation.

Plot 1 on tangent axis and construct two angles whose value of $\tan \boldsymbol{x}$ is 1. All angles are positive as domain is positive. Also no need to adjust domain as angle in equation is the same with angle for domain.

Than solutions are: $\boldsymbol{x} = \frac{\pi}{4}, \frac{5\pi}{4}$

11. $\sin \boldsymbol{x} \cos \boldsymbol{x} - \cos \boldsymbol{x} = 0 \qquad \boldsymbol{x} \in [0, 2\pi]$

Note; This equation looks similar to equation in example 10 but concept of solving it is completely different. The difference is coming from the fact that this equation is not possible to divide by $\cos \boldsymbol{x}$ because in this equation $\cos \boldsymbol{x} = 0$ and we can't divide by 0. In example 10, $\cos \boldsymbol{x}$ was different than 0 ($\cos \boldsymbol{x} \neq 0$) and we could divide. That is why always when we like to divide equation (any type) by some term or factor or any expression, we have to check if that is 0 or not. For example if we are solving polynomial equation $\boldsymbol{x}(\boldsymbol{x} - 2) = \boldsymbol{x}$ and if we want to divide by $\boldsymbol{x}$, we have to check if $\boldsymbol{x = 0}$ or not. Suppose that x = 0 and substitute in equation. Then we will have $0 \times (0 - 2) = 0$ and $0 = 0$ which is true statement and confirming $\boldsymbol{x = 0}$ and we can't divide. In example 10 we had $\sin \boldsymbol{x} = \cos \boldsymbol{x}$. Let check if $\cos \boldsymbol{x}$ equal to zero or not. If $\cos \boldsymbol{x} = 0$ than $\boldsymbol{x} = \frac{\pi}{2}$ and $\sin \frac{\pi}{2} = 1$ and we have statement $1 = 0$ which is not true statement and conclusion is that $\cos \boldsymbol{x} \neq 0$ and we can divide by $\cos \boldsymbol{x}$.

For example 11 we will get true statement 0 = 0 and that is why we can't divide by $\cos \boldsymbol{x}$. The above equation, then can be solved by null factor law,

$\sin \boldsymbol{x} \cos \boldsymbol{x} - \cos \boldsymbol{x} = 0$

$\cos \boldsymbol{x}(\sin \boldsymbol{x} - 1) = 0$

$\cos \boldsymbol{x} = 0$ or $\sin \boldsymbol{x} - 1 = 0$

$\sin \boldsymbol{x} = 1$

Solving each of those two equations we will get all solutions

$\boldsymbol{x} = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \boldsymbol{x} = \frac{\pi}{2}$

$\boldsymbol{x} = \frac{\pi}{2}, \frac{3\pi}{2}$

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APPLICATION OF DIFFERENTIATION

Practice 2

Applications of Differentiation – Practice 2

1)

2)

3)

4)

5)

6) Line $y = mx + k$ is a tangent to the curve with equation $f(x) = mx^2 + bx + c$ at $(2,-4)$. Find *m,b,c* and *k* If $f(x)$ has gradient -1 at $x = 3$.

7)

8)

9)

10)

11)

Solutions to Application of Differentiation – Practice 2

1)

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6) $y = mx + k$; $f(x) = mx^2 + bx + c$; $(2,-4)$; $x = 3 \rightarrow \text{grad.} -1$; $a,b,c,k \rightarrow ?$

We have to find four values (*a,b,c* and *k*). We have to identify four sets of data which describes those two lines and their relation, and for each set of data to form one equation for *a,b,c* and *k*. This concept (see explanation below) can be applied almost on any question of this or similar style.

1. Line $y = mx + k$ is a tangent to $f(x) = mx^2 + bx + c$ at $(2,-4)$. We know that derivative of $f(x) = mx^2 + bx + c$ is a gradient of a tangent at any point. At $x = 2$ the gradient of a tangent is *m*, then we can form one equation for *m,b* and *c*.

2. Point $(2,-4)$ is on the curve (on the line as well) $f(x) = mx^2 + bx + c$. That means that coordinates of this point satisfies the equation of this curve. By substituting 2 and -4 for *x* and $f(x)$ respectively in $f(x) = mx^2 + bx + c$, we will form second equation for *m, b* and *c*.

3. At $x = 3$ gradient (derivative of a curve) is -1. We can form third equation for *m, b* and *c* by substituting $x = 3$ in expression for derivative of $f(x) = mx^2 + bx + c$ and equalizing it with -1.

4. Line $y = mx + k$ passes through point $(2,-4)$. Since *k* is *y* intercept and we will have value for *m* by solving above described equations, then by using point $(2,-4)$ and value of m we can, simply, find *k* (*y* intercept).

$$f'(x) = 2mx + b$$

$$x = 2 \quad \rightarrow \quad f'(x) = m$$
$$2m \times 2 + b = m$$
$$3m + b = 0 \ldots\ldots\ldots\ldots (1)$$

$$(2,-4) \rightarrow \quad f(x) = mx^2 + bx + c$$
$$-4 = m \times 2^2 + b \times 2 + c$$
$$4m + 2b + c = -4 \ldots\ldots\ldots\ldots (2)$$

$$x = 3 \quad \rightarrow \quad f'(x) = -1$$
$$2m \times 3 + b = -1$$
$$6m + b = -1 \ldots\ldots\ldots\ldots (3)$$

Now we should solve those three equations because we need value of *m* that we can calculate *k*.

$$\left.\begin{aligned} &3m + b = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1) \\ &4m + 2b + c = -4 \ldots\ldots\ldots (2) \\ &6m + b = -1 \ldots\ldots\ldots\ldots\ldots (3) \end{aligned}\right\}$$

$(1) \quad \Rightarrow \quad b=-3m$

$b=-3m \rightarrow \quad 6m+b=-1$

$6m-3m=-1$

$3m=-1$

$\boxed{m=-\frac{1}{3}}$

$m=-\frac{1}{3} \rightarrow \quad b=-3m$

$b=-3\times\left(-\frac{1}{3}\right)$

$\boxed{b=1}$

$\left.\begin{array}{l} m=-\frac{1}{3} \\ b=1 \end{array}\right\} \rightarrow 4m+2b+c=-4$

$4\times\left(-\frac{1}{3}\right)+2\times1+c=-4$

$c=-6+\frac{4}{3}$

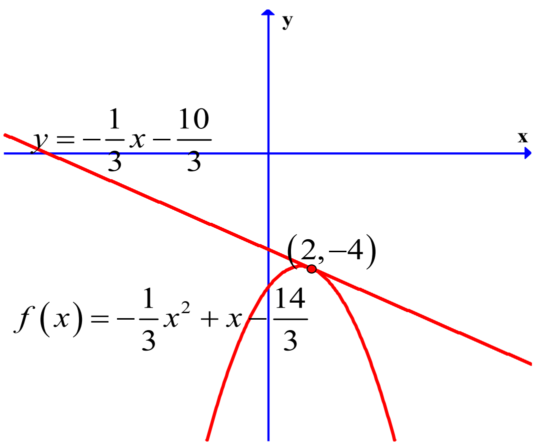
$\boxed{c=-\frac{14}{3}}$

Now we can find y intercept k.

$\left.\begin{array}{l} (2,-4) \\ m=-\frac{1}{3} \end{array}\right\} \rightarrow y=mx+k$

$-4=-\frac{1}{3}\times2+k$ and $\boxed{k=-\frac{10}{3}}$

Equations are $y=-\frac{1}{3}x-\frac{10}{3}$ and $f(x)=-\frac{1}{3}x^2+x-\frac{14}{3}$. Below is graphical illustration of this question.



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